Maths Learning Service: Revision **Mathematics IA**

Complex Numbers

The *imaginary number* $i = \sqrt{-1}$ is an extension to the real number system which allows us to solve equations such as

$$x^2 = -1.$$

A *complex number* is any number of the form $z = a + bi$, where a and b are real numbers.

Note: All numbers involving i can be written in this form.

Examples: (a) $i^2 + i^3$ (b) $\frac{2i-3}{i+1}$

(a)
$$\begin{aligned} & i^2 + i^3 \\ &= -1 + i^2 i \\ &= -1 - i \end{aligned}$$

(b)
$$\begin{aligned} & \frac{2i-3}{i+1} \\ &= \frac{2i-3}{i+1} \times \frac{i-1}{i-1} \\ &= \frac{2i^2-5i+3}{i^2-1} \\ &= \frac{-2-5i+3}{-1-1} \\ &= \frac{1-5i}{-2} = -\frac{1}{2} + \frac{5}{2}i \end{aligned}$$

Notes: (1) In $z = a + bi$, a is the *real part of* z.
b is the *imaginary part of* z.

(2) If $b = 0$, z is a real number.
If $a = 0$, z is a *purely imaginary number*.

$z = a - bi$ is the *complex conjugate* of $z = a + bi$. The solutions to quadratic equations are complex conjugates.

Example: Solve $x^2 - 2x + 10 = 0$.

Solution: Using the quadratic formula with $a = 1$, $b = -2$ and $c = 10$,

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 10}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{-36}}{2} \\ &= \frac{2 \pm \sqrt{36 \times -1}}{2} \\ &= \frac{2 \pm 6i}{2} \\ &= 1 - 3i \text{ or } 1 + 3i. \end{aligned}$$

Exercises

(1) Rewrite the following in the form $a + bi$:

(a) $(2i + 6) + (5i - 1)$ (b) $(3i + 2)(i - 1)$

(c) $\dfrac{i}{1 + 2i}$ (d) $\dfrac{1}{2 - i} + \dfrac{2}{2 + i}$

(2) If $z_1 = 1 - 2i$ and $z_2 = 2 + i$, find:

(a) $\overline{z_1} + \overline{z_2}$ (b) $\overline{z_1 + z_2}$

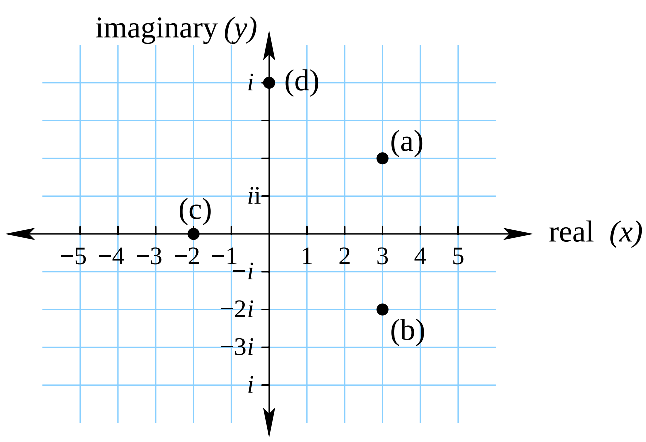
(3) Solve for x:

(a) $x^2 - 10x + 29 = 0$ (b) $\dfrac{1}{x} = 1 - 2x$

The Complex (Argand) Plane

The complex number $z = a + bi$ can be represented on a number *plane* (rather than a number *line*) with co-ordinates (a, b). The $x-$axis represents the real component of z and the $y-$axis represents the imaginary component.

Examples: (a) $3 + 2i$ (b) $3 - 2i$ (c) -2 (d) $4i$



The distance of a complex number from the origin of the Argand Plane is called the *modulus* of the complex number z (or $|z|$). By Pythagoras' Theorem:

$$|z| = \sqrt{a^2 + b^2}.$$

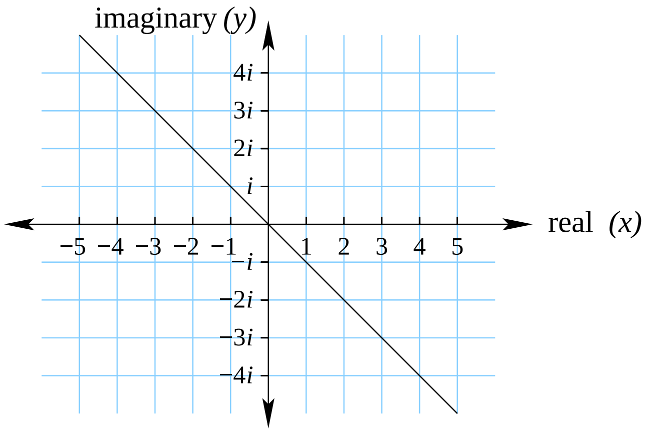
Examples: (a) $|3 + 2i|$ (b) $|3 - 2i|$ (c) $|-2|$ (d) $|4i|$

(a) $= \sqrt{3^2 + 2^2} = \sqrt{13}$

(b) $= \sqrt{3^2 + (-2)^2} = \sqrt{13}$

(c) $= \sqrt{(-2)^2 + 0} = 2$

(d) $= \sqrt{0 + 4^2} = 4$



(The reader is left to investigate why this solution works geometrically.)

Exercises

(4) Find:

(a) $|1 + i|$ (b) $|1 - i|$ (c) $|-6i|$

(5) Find all complex numbers $z = x + iy$ which satisfy:

(a) $|z + 1| = 1$ (b) $|z| = |z + 1|$

Answers to Exercises

(1) (a) $5 + 7i$ (b) $-5 - i$

(c) $\frac{2}{5} + \frac{1}{5}i$ (d) $\frac{6}{5} - \frac{1}{5}i$

(2) (a) $3 + i$ (b) $3 + i$

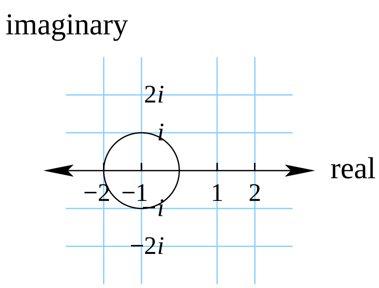
(This example demonstrates the property that $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$.)

(3) (a) $5 - 2i$ or $5 + 2i$ (b) $\frac{1}{4} - \frac{\sqrt{7}}{4}i$ or $\frac{1}{4} + \frac{\sqrt{7}}{4}i$

(4) (a) $\sqrt{2}$ (b) $\sqrt{2}$ (c) 6

(5) (a) $(x + 1)^2 + y^2 = 1$

(A circle of radius 1 centred on $(-1, 0)$)



(b) $x = -\frac{1}{2}$

(A vertical line)