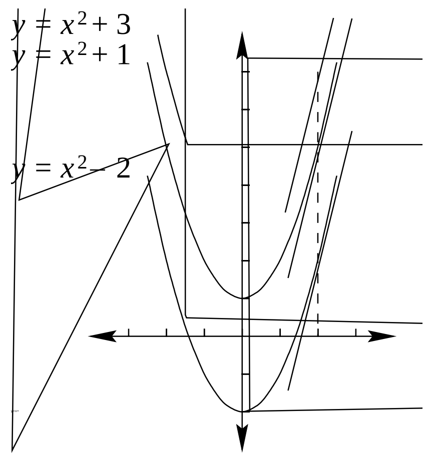
Anti-differentiation (Integration)

Anti-differentiation

Anti-differentiation or *integration* is the reverse process to differentiation. For example, if $f'(x) = 2x$, we know that this is the derivative of $f(x) = x^2$. Could there be any other possible answers?

If we shift the parabola $f(x) = x^2$ by sliding it up or down vertically, all the points on the curve will still have the same tangent slopes, i.e. derivatives. For example:



all have the same derivative function, $y' = 2x$, so a general expression for this family of curves would be

(3) $\int x^{-3}dx = -\frac{x^{-2}}{2} + c$ (Check: $\frac{d}{dx}\left(-\frac{x^{-2}}{2} + c\right) = -2 \times -\frac{x^{-3}}{2} + 0 = x^{-3}$ ✓)

(4) $\int x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{3/2} + c = \frac{2x^{\frac{3}{2}}}{3} + c$ (Check: $\frac{d}{dx}\left(\frac{2x^{\frac{3}{2}}}{3} + c\right) = \frac{3}{2} \times \frac{2x^{\frac{1}{2}}}{3} + 0 = x^{\frac{1}{2}}$ ✓)

Notice that a pattern emerges which can be summarized in mathematical notation as

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$$

for *any* real number n, except -1.

When $n = -1$ this formula would give $\int x^{-1}dx = \frac{x^0}{0} + c$, which is undefined. However, the integral does exist. Since $\frac{d}{dx}(\ln x) = \frac{1}{x}$ we can say $\int x^{-1}dx = \ln|x| + c$.

As a consequence of other basic rules of differentiation, we also have

$$\int kg(x)\, dx = k\int g(x)\, dx, \text{ where } k \text{ is a constant.}$$

$$\int (g(x) + h(x))\, dx = \int g(x)\, dx + \int h(x)\, dx$$

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + c, \text{ where } a, b \text{ and } c \text{ are constants.}$$

Examples: (1) $\int \left(x^2 + x\right) dx = \frac{x^3}{3} + \frac{x^2}{2} + c$ (2) $\int e^{3x-2}dx = \frac{1}{3}e^{3x-2} + c$

(3) $\int \left(4x^{\frac{1}{2}} + 3\right) dx = 4 \times \frac{2x^{\frac{3}{2}}}{3} + 3x + c = \frac{8x^{\frac{3}{2}}}{3} + 3x + c$

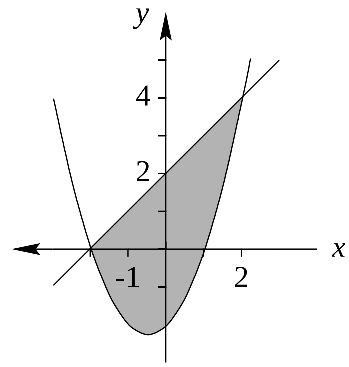
Exercises

1. Find the following integrals. Check each answer by differentiating.
   (a) $\int x^9 dx$ (b) $\int x^{\frac{1}{4}}dx$ (c) $\int x^{-5}dx$ (d) $\int x^{-\frac{3}{2}}dx$ (e) $\int 1\, dx$
2. Find the following integrals. Check each answer by differentiating.
   (a) $\int \Big(2x^{\frac{1}{2}}$ +94cmq[TF162rd[(3)TET100Tf2.5875tTd[(1)TET4e3.978Tf00Td[(2)TF1511.955Tf8.004-1.811

Definite Integration and areas under curves

The *definite integral* $\int_a^b f$

$$\int_{-2}^{2}\left(x+2-(x^2+x-2)\right)dx = \int_{-2}^{2}\left(4-x^2\right)dx = \left[4x-\frac{x^3}{3}\right]_{-2}^{2} = \left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right) = 16-\frac{16}{3} = \frac{32}{3}\text{ units}^2$$



Exercises

3. Calculate
   (a) $\int_{2}^{5} e^x\,dx$ (b) $\int_{-2}^{0} 3x^2\,dx$ (c) $\int_{4}^{9} 2\left(\sqrt{x}-x\right)dx$ (d) $\int_{-2}^{-1}\left(2x^3+\frac{1}{x^2}\right)dx$
4. Find the area between the following functions and the $x-$axis for the indicated interval.
   (a) $x^{\frac{1}{3}}$, $1 \le x \le 8$ (b) $\sqrt{x}-x$, $4 \le x \le 9$
5. Find the area between (a) $y = x - 2$ and $y = 2x - x^2$ (b) $y = x^2$ and $x = y^2$

ANSWERS

1. (a) $\frac{x^{10}}{10}+c$ (b) $\frac{4x^{\frac{5}{4}}}{5}+c$ (c) $-\frac{x^{-4}}{4}+c$ (d) $-2x^{-\frac{1}{2}}+c$ (e) $x+c$
2. (a) $\frac{4}{3}x^{\frac{3}{2}}-\frac{3}{x}+x+c$ (b) $x-2x^2+3x^3+c$ (c) $\frac{4}{3}x^3+2x^2+x+c$
   (d) $-\frac{3}{x}+c$ (e) $\frac{1}{7}e^{7x}+c$ (f) $-e^{-x-1}+c$
3. (a) 141.024 (b) 8 (c) $-39\frac{2}{3}$ (d) -7
4. (a) $11\frac{1}{4}$ (b) $19\frac{5}{6}$
5. (a) $4\frac{1}{2}$ (b) $\frac{1}{3}$