When giving matrices a name, use capital letters such as *A*, *B*, etc to distinguish them from

> *Matrix multiplication is only defined when the number of columns*
> *in the first matrix equals the number of rows in the second.*

(iv) $CD = \begin{matrix} 13 & 19 \\ 27 & 43 \end{matrix}$ but $DC = \begin{matrix} 16 & 22 \\ 27 & 40 \end{matrix}$ so $CD = DC.$

> *In general AB = BA for matrices.*

(v) $CI = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} = \begin{matrix} 1 \times 1 + 2 \times 0 & 1 \times 0 + 2 \times 1 \\ 3 \times 1 + 4 \times 0 & 3 \times 0 + 4 \times 1 \end{matrix}$

$= \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} = C$ (unchanged)

(vi) $IC = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} = C$ (unchanged)

> *The matrix I is an identity matrix and is the*
> *matrix equivalent of the number 1 in scalar multiplication.*

Notes: 1.   The identity is an exception to the general rule for matrix multiplication since $CI = IC = C$.

2.   Identity matrices only exist for square matrices. The matrix I used in Examples (v) and (vi) is called "the identity matrix for a 2×2 matrix". The identity matrix for a 3×3 matrix is $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$.

Exercises

$A = \begin{matrix} 1 & 0 \\ -1 & -2 \end{matrix}$ $B = \begin{matrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}$ $C = \begin{matrix} -1 \\ 2 \\ 2 \end{matrix}$ $D = \begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 4 \end{matrix}$ $E = \begin{matrix} -3 & 2 \\ 1 & 7 \end{matrix}$

(2) Using the above matrices, calculate the following (if possible):

(a) AB (b) BA (c) DI (d) ID (e) CD

(f) DC (g) BC (h) CB (i) E^2 (j) B^2

Inverse of a Square Matrix

In scalar algebra, we know that

$$a \times \frac{1}{a} = aa^{-1} = a^{-1}a = 1 \quad (a = 0).$$

We call a^{-1} the *multiplicative inverse* of a.

For square matrices, we define the inverse "A^{-1}" as having the property that

$$A \times A^{-1} = A^{-1} \times A = I.$$

The inverse of a 2×2 matrix is found by the formula below.

If $A = \begin{matrix} a & b \\ c & d \end{matrix}$ then

$$A^{-1} = \frac{1}{\det(A)} \begin{matrix} d & -b \\ -c & a \end{matrix}$$

where $\det(A) = |A| = $ *determinant* of $A = ad - bc.$

Notes: 1. A^{-1} can not be found by rearrangement ($A^{-1} = I \div A$), because division is not defined for matrices.

An application of the inverse: Solving Simultaneous Equations

A pair of simultaneous linear equations such as

$$-x + 2y = 0$$
$$x + y = 3$$

can be written in matrix notation as

$$\begin{matrix} -1 & 2 \\ 1 & 1 \end{matrix} \quad \begin{matrix} x \\ y \end{matrix} \quad = \quad \begin{matrix} 0 \\ 3 \end{matrix}$$

$$\text{or} \qquad A \qquad X \quad = \quad B.$$

If a unique solution exists, we can use the inverse matrix to solve the system, as follows:

$$AX = B$$
$$A^{-1}AX = A^{-1}B \qquad$$ Note that A^{-1} has been *pre*-multiplied on both sides. Since order of multiplication is important, we can't use BA^{-1} (i.e. *post*-multiplication) on the RHS since we pre-multiplied on the LHS.

$$IX = A^{-1}B \qquad \text{since } A^{-1}A = I$$
$$X = A^{-1}B \qquad \text{since } IX = X$$

In this example, we have $A = \begin{matrix} -1 & 2 \\ 1 & 1 \end{matrix}$ and hence $A^{-1} = -\dfrac{1}{3}\begin{matrix} 1 & -2 \\ -1 & -1 \end{matrix}$ so

$$X = A^{-1}B$$
$$= -\frac{1}{3}\begin{matrix} 1 & -2 \\ -1 & -1 \end{matrix} \begin{matrix} 0 \\ 3 \end{matrix}$$
$$= -\frac{1}{3}\begin{matrix} -6 \\ -3 \end{matrix}$$
$$= \begin{matrix} 2 \\ 1 \end{matrix}$$

Hence $x = 2$ and $y = 1$ is the answer (check by substituting back into the original equations).

Exercises

(5) Rewrite the following pairs of equations in the form of a matrix equation, $AX = B$, and solve (if a unique solution exists) using the inverse matrix of A.

(a) $\quad x - y = 5$
$\quad\quad x + y = 1$

(b) $\quad 5x + y = 7$
$\quad\quad 3x - 4y = 18$

(c) $\quad x + 2y = 8$
$\quad\quad 3x + 6y = 15$

(d) $\quad 2x + 3y = 11$
$\quad\quad 6x + 9y = 33$

Determinant of a 3×3 matrix

The inverse of a 3×3 matrix can be found using *row operations* (see revision sheet on Solving Linear Equations) but the determinant is as follows:

$$\text{If } A = \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \text{ then}$$

$$\det(A) = |A| = a \begin{array}{cc} e & f \\ h & i \end{array} - b \begin{array}{cc} d & f \\ g & i \end{array} + c \begin{array}{cc} d & e \\ g & h \end{array}.$$

This may seem like a complicated definition but the determinant can be thought of as a "first row expansion", where each entry in the first row is multiplied by the 2×2 determinant created by removing the row and column containing that entry. Notice also that the signs connecting the three terms alternate $(+, -, +)$.

Examples:

(i) $\begin{array}{ccc} 3 & 2 & 1 \\ 4 & 0 & 2 \\ 1 & 2 & 3 \end{array} = 3 \begin{array}{cc} 0 & 2 \\ 2 & 3 \end{array} - 2 \begin{array}{cc} 4 & 2 \\ 1 & 3 \end{array} + \begin{array}{cc} 4 & 0 \\ 1 & 2 \end{array} = 3(0-4) - 2(12-2) + (8-0) = -24$

(ii) $\begin{array}{ccc} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 2 & 2 & -2 \end{array} = -\begin{array}{cc} 0 & -2 \\ 2 & -2 \end{array} - 2 \begin{array}{cc} 3 & -2 \\ 2 & -2 \end{array} + 0 \begin{array}{cc} 3 & 0 \\ 2 & 2 \end{array} = -(0+4) - 2(-6+4) + 0 = 0$

Note: The matrix in Example (ii) has no inverse.

Exercises

(6) Find the following determinants:

(a) $\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{array}$ (b) $\begin{array}{ccc} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 4 & 2 & 2 \end{array}$ (c) $\begin{array}{ccc} -3 & 1 & -2 \\ 2 & 1 & 1 \\ 1 & -1 & -1 \end{array}$ (d) $\begin{array}{ccc} 3 & 3 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{array}$

(7) Show that the *upper triangular* matrix $\begin{array}{ccc} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{array}$ has determinant aei.

Answers to Exercises

(1)   (a) $\begin{matrix} 6 & 3 & -3 \\ 6 & 5 & 9 \end{matrix}$ (b) same as (a)   (c) $\begin{matrix} 12 & 7 \\ -4 & 7 \end{matrix}$ (d) $\begin{matrix} -10 & -3 \\ -4 & 11 \end{matrix}$

       (e) $\begin{matrix} 10 & 3 \\ 4 & -11 \end{matrix}$ (f) not possible   (g) not possible   (h) $\begin{matrix} -3 & 6 & 0 \\ 12 & 15 & 9 \end{matrix}$

       (i) $\begin{matrix} 13 & 9 \\ -8 & 16 \end{matrix}$ (j) not possible

(2)   (a) $\begin{matrix} 2 & 3 & 1 \\ -8 & -5 & -5 \end{matrix}$ (b) not possible   (c) D (d) D

       (e) not possible   (f) $\begin{matrix} 3 \\ 6 \\ 15 \end{matrix}$ (g) $\begin{matrix} 6 \\ 3 \end{matrix}$ (h) not possible

       (i) $E^2 = EE = \begin{matrix} 11 & 8 \\ 4 & 51 \end{matrix}$ (j) not possible

(4)   (a) $\dfrac{1}{20} \begin{matrix} 8 & -18 \\ -2 & 7 \end{matrix}$ or $\begin{matrix} \frac{2}{5} & -\frac{9}{10} \\ -\frac{1}{10} & \frac{7}{20} \end{matrix}$ (b) $\dfrac{1}{4} \begin{matrix} 2 & -2 \\ -1 & 3 \end{matrix}$

       (c) determinant = 0 so no inverse exists   (d) $-\dfrac{1}{14''} \begin{matrix} 4 & -6 \\ -3 & 1 \end{matrix}$

       1

       (e) -13