This Topic . . .

This Topic begins by introducing the gradient of a curve. This concept was invented by Pierre de Fermat in the 1630s and made rigorous by Sir Issac Newton and Gottfried Wilhelm von Leibniz in the 1670s.

The process of nding the gradient by algebra is called *di erentiation*. It is a powerful mathematical technique and many scienti c discoveries of the past three centuries would have been impossible without it. Newton used these ideas to discover the Law of Gravity and to nd equations describing the orbits of the planets around the sun.

Di erentiation remains a powerful technique today and has many theoretical and practical applications.

The Topic has 2 chapters:

Chapter 1 explores the rate at which quantities change. It introduces the gradient of a curve and the rate of change of a function. Examples include motion and population growth.

Chapter 2 introduces derivatives and di erentiation. Derivatives are initially found from *rst principles* using limits. They are then constructed from known results using the *rules of di erentiation* for addition, subtraction, multiples, products, quotients and composite functions. Implicit di erentiation is also introduced. Applications include nding tangents and normals to curves.

Auhor: Dr Paul Andrew

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Chapter 1

Gradients of Curves

1.1 Describing change

How can we describe the rate at which quantities change?

Example

constant velocity

gradient of a line

The *distance-time graph* below shows the distance travelled by a car that has a constant velocity of 15 m=s.[1]

Velocity measures how distance changes with time. You can see that the car travelled 15m in 1 second, 30m in 2 seconds, etc. When the velocity is constant, it is calculated using the ratio:[2]

$$\frac{\text{distance}}{\text{time}} = \frac{\text{change in distance}}{\text{change in time}}$$

Questions like these motivated Fermat, Newton and Leibniz in their exploration of the gradient of a curve and led to the invention of di erentiation.

1.2 Gradients of curves

How can we determine the vertical velocity of an experimental rocket from its height-time graph? What is its velocity at *exactly* $t = 2$ seconds?

The second question can be approached by estimating the velocity *near* $t = 2$:

> at $t = 2$, the rocket's height is $78{:}4\ m$, and at $t = 3$ the height is h $= 102{:}9\ m$, so the vetrical velocity between $t = 2$ and $t = 3$ is approximately:
>
> $$\frac{\text{height}}{\text{time}} = \frac{\text{change in height}}{\text{change in time}} = \frac{102{:}9 \quad 78{:}4}{3 \quad 2} = 24{:}5\ m{=}s$$
>
> . . . the velocity at t = 2 is about 24.5 $m{=}s$.
>
> at $t = 2{:}5$ the height is $91{:}8\ m$,
> so the vertical velocity between $t = 2$ and $t = 2{:}5$ is approximately:
>
> $$\frac{\text{height}}{\text{height}} = \frac{\text{change in height}}{\text{change in height}} = \frac{91{:}8 \quad 78{:}4}{2{:}5 \quad 2} = 26{:}8\ m{=}s$$
>
> . . . so 26.8 $m{=}s$ is a better estimate of the velocity at $t = 2$.

Smaller time intervals will give better estimates of the velocity at $t = 2$.

These estimates can be interpreted as gradients:

The first estimate of the velocity was 24.5 m/s.

. . . it is the gradient of the line from $(2, 78.4)$ *to* $(3, 102.9)$.

The second estimate was 26.8 m/s.

. . . it is the gradient of the line from $(2, 78.4)$ *to* $(2.5, 91.8)$.

The third estimate was 30 m/s.

. . . it is the gradient of the line from $(2, 78.4)$ *to* $(2.01, 78.7)$*, and is very close to the gradient of the line that just touches the curve at* $(2, 78.4)$.

The velocity of the rocket at $t = 2$ is equal to the gradient of the straight line that just touches the curve at $t = 2$.

Terminology

A straight line that just touches a curve is called a *tangent line*.

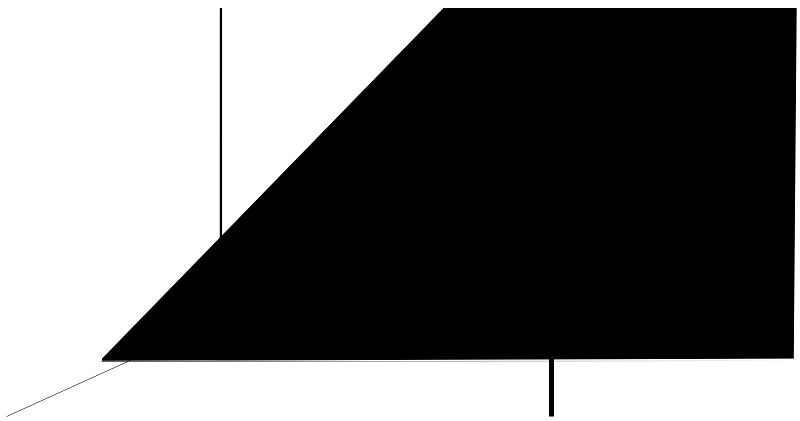
The *gradient of a curve* at a point is the gradient of the tangent line to the curve at that point.

> *The gradient of the curve at a point (or the gradient of the tangent line) measures the rate of change of the quantity at the point.*

Example

changing growth rate

The growth rate of an aphid population can be found from the gradient of the *population-time* graph. This observation is used to describe how the population changes over 60 days.



In the graph . . .

the gradient of the curve incrwtho-291(t3-326(of)-3=(of)-30(t3-326o(the)-3(of)-3=(of)7(3

the gradient of the curve is positive but reducing from $t = 0$ to $t = 5$, so
. . . *the velocity is positive (climbing) and decreasing for* $0 \quad t < 5$.

the gradient of the curve is zero at $t = 5$, so
. . . *the (vertical) velocity is zero at* $t = 5$.

the gradient of the curve is negative and growing for $5 < t \quad 10$, so
. . . *the velocity is negative (returning) and increasing for* $0 \quad t < 5$.

Example

vertical
velocity &
acceleration11.955 Td [(ac)51(c-11.955 Td [(velo)51(city)-358(&)]TJ -7.511 -11.955 35)-350(90 g e955 TdHerng)-326(from)]T(/

Exercise 1.2

1. Sketch the graph of $y = x^2$ for $0 \quad x \quad 2$, then draw chords[5] from $P(1; 1)$ to the points $Q(0{:}5; 0{:}25)$ and $R(1{:}5; 2{:}25)$.

   (a) What are the gradients of the chords

      i. PQ

      ii. PR

   (b) Show the tangent to $y = x^2$ at $x = 1$ has gradient between 1.5 and 2.5 .

   (c) By selecting other chords on $y = x^2$

Exercise 1.3

1. Sketch the graph of $y = x^2$ for $0 \le x \le 4$ and draw the chords from $P(2, 4)$ to the points $Q(1, 1)$, $R(3, 9)$ and $S(4, 16)$.
2. What is the average rate of change of x^2 between
   (a) $x = 1$ and $x = 2$
   (b) $x = 2$ and $x = 3$
   (c) $x = 2$ and $x = 4$
3. Use your answer to question 2 to deduce that the rate of change of x^2 with respect to x at $x = 2$ is between 3 and 5.
4. Estimate the rate of change of x^2 with respect to x at $x = 2$ to within ± 0.1.
5. The graph of $y = x^2 + 1$ can be obtained by shifting the graph of $y = x^2$ upwards by one unit. Use this together with your answer to question 4 to estimate the instantaneous rate of change of $x^2 + 1$ with respect to x at $x = 2$.

Chapter 2

Differentiation

The *gradient of a curve* shows the rate at which a quantity changes on a graph.

If the quantity is described by a function[1] then the rate of change of the function can be found directly by algebra without drawing a graph. This process is called *differentiation*. We call the rate of change of a function the *derivative of the function.*

There are two ways of finding derivatives of functions:

- from first principles, following the footsteps of the early mathematicians. This is mainly of historical interest, however it introduces the important idea of a *limit* and explains the notation used in differentiation.
- constructing derivatives from known results using *the rules of differentiation*.

2.1 From first principles ...

The gradient at a point on a curve can be found exactly by algebra when the equation of the curve is known.

This section shows how the early mathematical explorers calculated gradients and derivatives.

Example

from first principles

To find the gradient of the parabola $y = x^2$ at the point $P(1; 1)$:

1. Find the gradient of the line through $P(1; 1)$ and a second point Q on the parabola. (See diagram on page 12.)

   We take the second point to be $Q(1 + h; (1 + h)^2)$ *where* h *stands for a number.*[2]

Q

P

You can see that when h is a very small number, the line through the points P and Q will be very close to the tangent line at (1, 1).

2. The gradient of the line from P (1; 1) to Q(1 + h; (1 + h)^2) is

$$
\begin{aligned}
\frac{y}{x} &= \frac{(1+h)^2 \quad 1}{1+h \quad 1} \\
&= \frac{(1+2h+h^2) \quad 1}{1+h \quad 1} \\
&= \frac{2h+h^2}{h} \\
&= \frac{h(2+h)}{h} \\
&= 2+h \qquad \ldots \text{as long as } h \neq 0.
\end{aligned}
$$

3. When h is very small, the gradient

$$\frac{y}{x} = 2 + h$$

will be very close to the gradient of the tangent line at (1, 1).

We can deduce that ....

as h becomes smaller and smaller, $\frac{y}{x}$ becomes closer to 2.

x^2

This shows that the gradient of the parabola $y = x^2$ at (1

Example

gradient of $y = x^2$ at $x = a$

To nd the gradient of the parabola $y = x^2$ at $x = a$ means to nd the gradient at the point $(a; a^2)$ on the parabola:

1. Find the gradient of the line through $P(a; a^2)$ and a second point Q on the parabola.

   We take the second point to be $Q(a + h; (a + h)^2)$ *where* h *stands for a number.*[3]

   Q

   P

   When h is a very small number, the line going through the points P and Q will be very close to the tangent line at $(a; a^2)$.

2. The gradient of the line from $P(a; a^2)$ to $Q(a + h; (a + h)^2)$ is

$$
\begin{aligned}
\frac{y}{x} &= \frac{(a+h)^2 \quad a^2}{a+h \quad a} \\
&= \frac{(a^2 + 2ah + h^2) \quad a^2}{a+h \quad a} \\
&= \frac{2ah + h^2}{h} \\
&= \frac{h(2a+h)}{h} \\
&= 2a + h \qquad \ldots \text{ as long as } h \neq 0
\end{aligned}
$$

3. When h is very small, the gradient

$$\frac{y}{x} = 2a + h$$

   will be very close to the gradient of the tangent line at $(a; a^2)$.

   We can deduce that ....

   as h becomes smaller and smaller, $\frac{y}{x}$ becomes closer to 2a.

[3] If $x = a + h$, then $y = x^2 = (a + h)^2$.

we just write

$$\frac{dy}{dx} = 2x \qquad \text{or} \qquad \frac{d}{dx}(x^2) = 2x$$

These are read aloud as

dee y dee x equals 2*x* or *dee dee x of x squared equals* 2*x*

This notation is adjusted when di erent variables are used. For example, if the relationship between population (P) and time (t) is given by $P = t^2$, then the population growth rate is $2t$ and we can write either

$$\frac{dP}{dt} = 2t \qquad \text{or} \qquad \frac{d}{dt}(t^2) = 2t$$

(d) These traditional symbols are clumsy to type without special software and are often replaced by dashes. For example, we can write

y^0 or $y^0(x)$ instead of $\frac{dy}{dx\,y}$

Exercise 2.1

1. Find the derivative of $y = x^2$ at $x = 3$ from first principles by:
   (a) sketching the parabola $y = x^2$
   (b) marking the points $R(3; 9)$ and $S(3 + h; (3 + h)^2)$ on it, where h is some number.
   (c) finding the gradient $\frac{\Delta y}{\Delta x}$ of the line RS
   (d) evaluating the limit $\lim_{h \to 0} \frac{\Delta y}{\Delta x}$
2. Find the derivative of $y = x^2 - 2x$ at $x = 2$ from first principles by:
   (a) sketching the parabola $y = x^2 - 2x$
   (b) marking the points $U(2; 0)$ and $V(2 + h; (2 + h)^2 - 2(2 + h))$ on it, where h is some number.
   (c) finding the gradient $\frac{\Delta y}{\Delta x}$ of the line UV
   (d) evaluating the limit $\lim_{h \to 0} \frac{\Delta y}{\Delta x}$
3. If $y = x^2 - 2x$, then it is known that $\frac{dy}{dx} = 2x - 2$. Use this to:
   (a) find the gradient of the parabola at the y-intercept
   (b) find the equation of the tangent line at the y-intercept
4. A fish population is increasing according to the quadratic model[4]

$$P(t) = 600t - t^2 \text{ fish/day.}$$

   (a) Sketch this model for $0 \leq t \leq 600$.

   If $P'(t) = 600 - 2t$:

   (b) find the population growth rate when $t = 100$
   (c) find when the population growth rate is zero
   (d) find the maximum size of the population

[4] formulae representing real life situations are frequently called *models*.

2.2 Constructing derivatives . . .

The early mathematicians discovered that

There are two powers that occur frequently and whose derivatives are worth memorising: $1\,(= x^0)$ and $x\,(= x^1)$.

If $y = 1\,(= x^0)$, then $\frac{dy}{}$ x

2.2.2 Polynomials

Many mathematical functions are built from simpler functions such as powers. We use this to construct their derivatives.

The general form of a *polynomial* of degree n in x is

$$ax^n + bx^{n-1} + \dots + dx + e\,,$$

where a, b, . . . , d, e are numbers. It is constructed by adding or subtracting multiples of powers of a single variable (x in this case), and a constant term.

It can be differentiated by using the following rules:

Rule 1 (constants)
The derivative of a constant is zero.

$$f(x) = c \Rightarrow f'(x) = 0$$

Rule 2 (multiples)
The derivative of a constant multiple is the multiple of the derivative.

$$y = cf(x) \Rightarrow y' = cf'(x)$$

Rule 3 (sums)
The derivative of a sum of terms is the sum of their derivatives.

$$y = f(x) + g(x) + \dots \Rightarrow y' = f'(x) + g'(x) + \dots$$

Example

applying the rules for differentiation

1. If $y = 100x$

Rule 3 can also be applied to to differences.[6] This is because a difference such as $f(x) - g(x)$ can be rewritten as the sum of $f(x)$ and $(-1)g(x)$. We don't bother to write down every detail when differentiating, but take it for granted that Rule 2 and 3 imply that:

the derivative of a sum (or difference) of terms is the sum (or difference) of their derivatives.

Example

derivatives of differences

1. If $y = -x^2 + 2x + 3$, then $y' = -2x + 2$.
2. If $y = x^2 - 2x + 3$, then $y' = 2x - 2$.
3. If $y = x^2 + 2x - 3$, then $y' = 2x + 2$.
4. If $y = (x + 3)(x - 7)$, then

$$\begin{aligned} y &= (x+3)(x-7) \\ &= x^2 - 4x - 21 \\ y' &= 2x - 4 \end{aligned}$$

Example

vertical velocity & maximum height

The height h (metres) of an experimental rocket after t seconds is given by

$$h = 49t - 4.9t^2 \ m/s.$$

2x θ 4 X 2Tg [4

The vertical velocity (rate of change of height with time) is

dh

The rocket reaches its maximum height when its vertical velocity is 0 m/s. This occurs when

$$\frac{dh}{dt} = 49 - 9.8t = 0 \Rightarrow t = \frac{49}{9.8} = 5\,s$$

2.2.3 Products and quotients

When functions are built from the products and quotients of simpler functions, we can use the following rules for constructing their derivatives.

> **Rule 4** (products)
> The derivative of a product is the derivative of the rst function multiplied by the second function, plus the rst function multiplied by the derivative of the second function.
>
> $$y = f(x)g(x) \;=)\; y^0 = f^0(x)g(x) + f(x)g^0(x)$$

Example

products
$f^0 g + f g^0$

(a) If $y = (x + 1)(x^2 + 2)$, then

$$\begin{aligned} y^0 &= 1 \quad (x^2 + 2) + (x + 1) \quad 2x \\ &= 3x^2 + 2x + 3 \end{aligned}$$

(b) If $f(x) = 15 \quad 3(x + 1)(x^2 + 2)$, then

$$\begin{aligned} f^0(x) &= 0 \quad 3[1 \quad (x^2 + 2) + (x + 1) \quad 2x] \\ &= \quad 3(x^2 + 2x + 3) \end{aligned}$$

Exercise 2.2.3

1. Use the product rule to di erentiate the following

   (a) $y = x^2(2x \quad 1)$

   (b) $y = (x + 1)(x^3 + 3)$

   (c) $y = (x^3 + 6x^2)(x^2 \quad 1) + 20$

   (d) $u = (7x + 3)(2 \quad 3x) + (x$ + 35 11.9552 T7m]TJ/F25 11.9552 Tf 10.406 0 01.9552 Tf 9.30

Rule 5 (quotients)
The derivative of a quotient is the derivative of the numerator multiplied by the denominator, less the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator.

$$y = \frac{f(x)}{g(x)} =$$

2.2.4 Composite functions and the chain rule

A function like $y = (x^2 + 1)^{50}$

Exercise 2.2.4

1. If $f(x) = x^2 - 3$ and $g(x) = x^5$, find
   (a) $f(g(x))$ or $f \circ g(x)$
   (b) $g(f(x))$ or $g \circ f(x)$
2. If $f(x) = 3x + 2$ and $g(x) = \sqrt{x}$, find
   (a) $f(g(x))$
   (b) $g(f(x))$
3. If $h(x) = 2x^2 + 1$ and $j(x) = x^3$, find
   (a) $h \circ j(x)$
   (b) $j \circ h(x)$
4. If $l(x) = x^2$ and $m(x) = \frac{1}{2}x$, find
   (a) $l(m(x))$
   (b) $m(l(x))$
5. Identify outside and inside functions for the composite functions below.[7]
   (a) $(x + 1)^5$
   (b) $\sqrt{x - 4}$
   (c) $(x^2 - 3x + 4)^2$
   (d) $(3x + \sqrt{x})^3$
6. If $f(x) = x^2$, $g(x) = x + 1$ and $h(x) = 2x$, find $f(g(h(x)))$ or $f \circ g \circ h(x)$.

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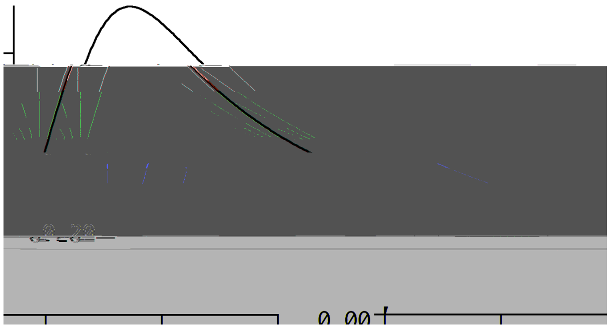
[7] When there is more than one answer, choose the inside and outside functions that are simplest.

Rule 6

9. Use the product, quotient and chain rules to differentiate:

   (a) $x\sqrt{x+1}$ (b) $x^2\sqrt{2x+3}$ (c) $\dfrac{x}{\sqrt{3x+4}}$ (d) $\dfrac{x^2}{\sqrt{x^2+1}}$

10. The graph of $y = \dfrac{x}{\sqrt{x^4+1}}$ is shown below for $x \geq 0$.



   (a) Find $\dfrac{dy}{dx}$

   (b) Solve $\dfrac{dy}{dx} = 0$

   (c) Find the maximum value of $\dfrac{x}{\sqrt{x^4+1}}$ for $x \geq 0$.

2.2.5 Implicit Differentiation

When functions are *explicitly* defined in the form $y = f(x)$ they can be differentiated using the previous rules of differentiation. However functions can also be *implicitly* defined.

Example

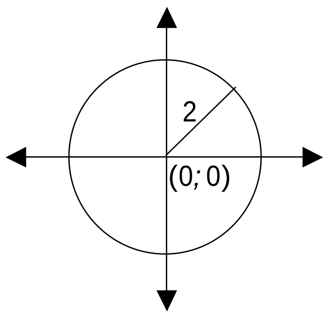
implicit relationship

Consider the equation of the circle

$$x^2 + y^2 = 4$$

with centre $(0, 0)$ and radius 2.

This is an example of an *implicit* relationship between x and y.



Solving this relationship for y gives y *explicitly* in terms of x, that is as the subject of a formula with x as the independent variable.

$$\begin{aligned} x^2 + y^2 &= 4 \\ y^2 &= 4 - x^2 \\ y &= \sqrt{4 - x^2} \end{aligned}$$

The gradient of the tangent line to $x^2 + y^2 = 4$ at $(\sqrt{2}, \sqrt{2})$, can now be found by differentiating $y = \sqrt{4 - x^2}$, giving ...

$$\frac{dy}{dx} = -\frac{x}{\sqrt{4 - x^2}} = -\frac{\sqrt{2}}{\sqrt{4 - (\sqrt{2})^2}} = -1$$

It may be difficult or impossible to solve an implicit relationship between x and y in such a way as to make y the subject of a formula with x as the independent variable. In these cases we use the technique of *implicit differentiation* to find $\dfrac{dy}{dx}$.

Example

implicit differentiation

To find $\dfrac{dy}{dx}$ directly from the implicit relationship $x^2 + y^2 = 4$ ...

1. Assume that y is a function of x, writing it as $y(x)$.

2. Differentiate both sides of $x^2 + y^2 = 4$, e.g.

$$\begin{aligned}
x^2 + y^2 &= 4 \\
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(4) \\
2x + \frac{d}{dx}(y^2) &= 0 \qquad \text{. . . differentiating each term} \\
2x + 2y\frac{dy}{dx} &= 0 \qquad \text{. . . by the chain rule} \\
2y\frac{dy}{dx} &= \quad 2x \\
\frac{dy}{dx} &= \quad \frac{x}{y} \qquad \text{. . . provided } y \neq 0
\end{aligned}$$

The derivative can now be used to find the gradient of the tangent line at any point on $x^2 + y^2 = 4$. For example, at the point $(\sqrt{2}, \sqrt{2})$:

$$\frac{dy}{dx}$$

The equation of the tangent line is:

$$y = \quad x + c; \text{ for some number } c.$$

Substituting $(1; \quad 1)$ into this equation shows that $c = 2$, and that the equation of the tangent line at $(1; \quad 1)$ is $y = \quad x + 2$.

> A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.
>
> If the gradient of the tangent line is $m_1 (\neq 0)$, and the gradient of the normal is m_2, then
>
> $$m_1 m_2 = \quad 1.$$

Example

Substituting $(\sqrt{2}, 2\sqrt{2})$ into this equation shows that

$$c = \frac{3\sqrt{2}}{2};$$

and that the equation of the tangent line at $(\sqrt{2}, 2\sqrt{2})$ is $y = 2x + \frac{3\sqrt{2}}{2}$.

Exercise 2.2.5

1. Find $\frac{dy}{dx}$ if:

   (a) $x^2 + y^2 = 16$

   (b) $x^2 + 3y^2 = 9$

   (c) $x^2 - y^2 = 25$

   (d) $x^2 + xy + y^2 = 10$

   (e) $x^3 + 2x^2y + y^2 = 10$

2. Find the gradient of the tangent line to:

   (a) $x^2 + y^2 = 1$ at $(\sqrt{2}, \sqrt{2})$

   (b) $x^2 - xy + y^2 = 1$ at $(1, 1)$

   (c) $x + y = 2xy$ at $(1, 1)$

3. Find the equation of the normal to:

   (a) $x^2 + y^2 = 8$ at $(2, 2)$

   (b) $x^2 + \frac{y^2}{2} = 3$ at $(1, 2)$

4. Show that the normal to the circle $x^2 + y^2 = 1$ at the point (a, b) with $ab \neq 0$ always passes through the origin.

Appendix A

First Principles

The graph below shows how a function $f(x)$ might change between $x = a$ and $x = b$.

The gradient of the chord from $(a; f(a))$ to $(b; f(b))$ is

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}:$$

As the width of the interval $[a; b]$ decreases, the approximation

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}:$$

becomes closer to *the gradient of the tangent line* to the graph of $f(x)$ at $x = a$, and so to *the derivative* of $f(x)$ at $x = a$.

If we put $b = a + h$, then the derivative of $f(x)$ at $x = a$ is given by the limit

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{(a + h) - a}$$

Definition

The derivative of $y = f(x)$ at the point $(a; f(a))$ is given by the limit

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example

first principles at $x = a$

Find the derivative of $y = x^2$ at $x = a$ using first principles

1. From the definition ...

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{(a^2 + 2ah + h^2) - a^2}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2}{h} \\ &= \lim_{h \to 0} 2a + h \\ &= 2a \end{aligned}$$

The derivative at $x = a$ is $\frac{dy}{dx} = 2a$

Example

first principles without using $x = a$

Differentiate $y = x^2 + 4x + 2$ using first principles

1. From the definition ...

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[(x+h)^2 + 4(x+h) + 2] - [x^2 + 4x + 2]}{h}$$

2. Expanding, then simplifying and taking the limit ...

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{[x^2 + (2h+4)x + (h^2 + 4h + 2)] - [x^2 + 4x + 2]}{h} \\ &= \lim_{h \to 0} \frac{2hx + (h^2 + 4h)}{h} \\ &= \lim_{h \to 0} 2x + h + 4 \\ &= 2x + 4 \end{aligned}$$

The derivative is $\frac{dy}{dx} = 2x + 4$

Example

cubic function

Differentiate $y = x^3$ using first principles

1. From the definition . . .

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

2.

Exercise A

1. Use rst principles to nd the derivative of $y = x^2 \quad x$ at $x = a$.
2. Di erentiate $y = x^4$ using rst principles.
3. Di erentiate $f(x) = \dfrac{x \quad 1}{x \quad 2}$ using rst principles.

Appendix B

Answers

Exercise 1.1

1(a)

1(b) The constant velocity is 30m=s.

1(c)

1(d) The constant acceleration is 0m=s^2.

Exercise 1.2

1(a) (i) $m_{PQ} = \dfrac{1 \quad 0{:}25}{1 \quad 0{:}5} = 1{:}5$ (ii) $m_{PR} = \dfrac{2{:}25 \quad 1}{1{:}5 \quad 1} = 2{:}5$

1(b) This follows from $m_{PQ} < m_{tangent} < m_{PR}$.

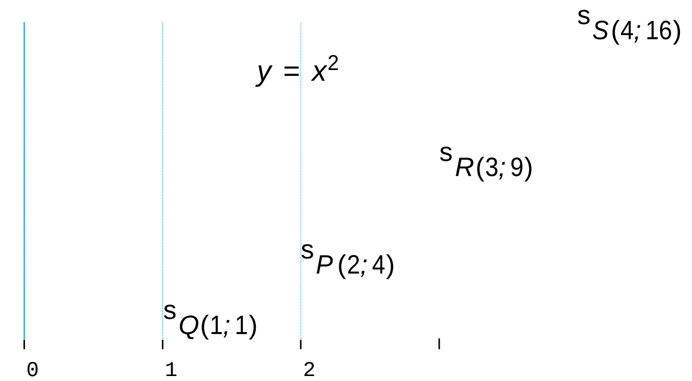
1(c) Using the points $L(0{:}9; 0{:}81)$ and $M(1{:}1; 1{:}21)$, $m_{tangent}$ 2 is an estimate to within 0:1 as

$$m_{PL} = 1{:}9 < m_{tangent} < m_{PM} = 2{:}1$$

2. (i) matches (a), (ii) matches (c), (iii) matches (d), (iv) matches (b)

Exercise 1.3

1.



2(a) $m_{QP} = 3$ 2(b) $m_{PR} = 5$ 2(c) $m_{PS} = 6$

3. As $m_{QP} \quad m_{tangent} \quad m_{PR}$

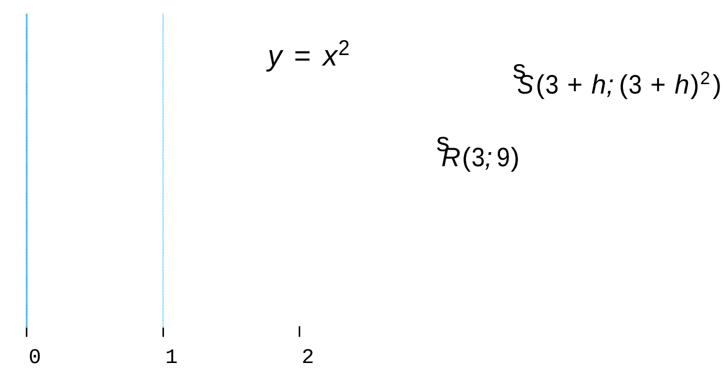
4. Using the points $L(1{:}9; 0{:}81)$ and $M(2{:}1; 1{:}21)$, $m_{tangent}$ 4 is an estimate to within   0:1 as

$$m_{PL} = 3{:}9 < m_{tangent} < m_{PM} = 4{:}1$$

5. When the graph is shifted the new tangent line remains parallel to the old, and so has the same gradient.

Exercise 2.1

1(a) & 1(b)



1(c) & 1(d)

$$\frac{y}{x} = \frac{(3+h)^2 \quad 9}{(3+h) \quad 3}$$

$$= \frac{6h + h^2}{h}$$

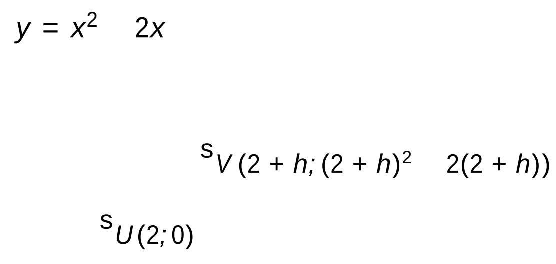
$$= 6 + h$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y}{x}$$

$$= \lim_{h \to 0} 6 + h$$

$$= 6$$

2(a) & 2(b)



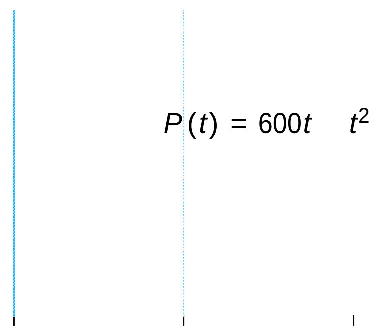
2(c) & 2(d)

$$\frac{y}{x} = \frac{[(2+h)^2 \quad 2(2+h)] \quad 0}{(2+h) \quad 2}$$
$$= \frac{2h + h^2}{h}$$
$$= 2 + h$$

$$\frac{dy}{dx} = \lim_{h! \ 0} \frac{y}{x}$$
$$= \lim_{h! \ 0} 2 + h$$
$$= 2$$

3(a) The y-intercept is $(0; 0)$. The gradient is $m = \quad 2$.
3(b) The tangent line is $y = \quad 2x$.

4(a)



4(b) The population growth rate is $P^0(100) = 600 \quad 2 \quad 100 = 400$.
4(c) Solving $P^0(t) = 0$ gives $t = 300$ days.
4(d) This occurs when $P^0(t) = 0$ at $t = 300$. The maximum population size is $P(300) = 90; 000$.

Exercise 2.2.1

1(a) $\dfrac{dy}{dx} = 20x^{19}$ 1(b) $\dfrac{dy}{dx} = \dfrac{2}{x^3}$ 1(c) $\dfrac{dv}{du} = 3u^2$ 1(d) $\dfrac{dv}{du} = \dfrac{1}{2u\sqrt{u}}$

2. As $h'(t) = 4t^3$, $h'(2) = 32$.

Exercise 2.2.4

1(a) f