MathsStart

This Topic...

This topic

1
Exponential Equations & Logarithms

Example

2 on a calculator: use $\boxed{\ln}$ 2 $\boxed{=}$ 0.6931

d the value of log *2* on a calculator: use $\boxed{\log}$ 2 $\boxed{=}$ 0.3010

Example

Solve the equation $2e^{4x} = 3$

Answer

Example

Solve the equation $(e^x)^2 = 3$

Answer 1

Take square roots of both sides:

$$\begin{array}{rl} (e^x)^2 & 3 \\ e^x & \sqrt{3} \\ x & \ln(\sqrt{3}) \\ & 0.5493 \end{array}$$

Answer 2

Use index rule 3 from *Topic 7*: $(a^n)^m = a^{nm}$

$$\begin{array}{rl} (e^x)^2 & 3 \\ e^{2x} & 3 \\ 2x & \ln 3 \\ x & \dfrac{\ln 3}{2} \\ & 0.5493 \end{array}$$

Problems 1.2

1. Solve the following equations:

(a) $e^x = 1$ (b) $e^x = 2$ (c) $e^x = 3$ (d) $e^x = 2^2$ (e) $e^x = 3^2$

(f) $10^x = 1$ (g) $10^x = 2$ (h) $10^x = 2^2$ (i) $e^x = 2\ \ ^2$ (j) $e^x = \sqrt{\ }$

2. Solve the following equations:

(a) $e^x \quad 2 = 0$ (b) $2e^x \quad 5 = 0$ (c) $3e^x = 2 + 2e^x$ (d) ———

(e) $10^{2x} = 1$ (f) $10^{2x} = 2$ (g) $10^{2x} = 2^2$ (h) ———

3. Use the index rules to solve the following equations.

(a) $e^x.e^{2x}.e^{3x} = 30$ (b) $(e^{2x})^3 = 10$ (c) $\sqrt{\ }$ (d) $(e^{3x})^4 = 10e^{2x}$

4. If the population of the earth was 6.5 billion in 2000, and was increasing at 2% per year,

(a) what would the population be in 20 years time?

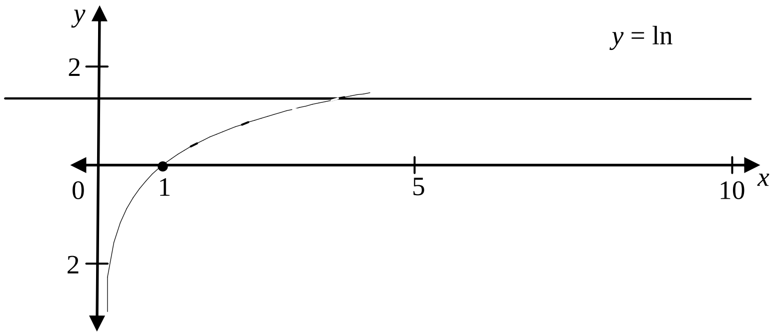
(b) when would the population reach 15 billion?

2

Logarithm Functions

2.1 The Natural Logarithm Function and its Graph

The equation $e^y = x$ has a solution $y = \ln x$ for every positive value of x, so the natural domain of $\ln x$ is $\{x : x > 0\}$. The graph is show below.



x-intercept is (1, 0) because $e^0 = 1$ ln 1 = 0.

x-axis when $x > 1$

x-axis when $0 < x < 1$.

$y = e^x$ and $y = \ln x$ are related. The diagram below shows that one is the reflection of the other across the line $y = x$.

The reason for this is that $e^r = s$ $r = \ln s$.

However, $e^r = s$ means that the point (r, s) is on the curve $y = e^x$, and $r = \ln s$ means that the point (s, r) is on the curve $y = \ln x$. This means that (r, s) is on the curve $y = e^x$ whenever (s, r) is on the curve $y = \ln x$. For example, $(0, 1)$ is on $y = e^x$ and $(1, 0)$ is on $y = \ln x$. This is another way of saying that the curves

Problems 2.2

1. Express each of the following as a single logarithm.

(a) $\ln 6 + \ln 3$ (b) $\ln 56 \quad \ln 7$ (c) $2 \ln 5$ (d) $\ln 5 + 2 \ln 2$

(e) $2 \ln 10 \quad 2 \ln 2$ (f) $1 \quad \ln (2e)$ (g) $2 + \ln 3$ (h) 0

2. Solve the following equations

(a) $\ln (x \quad 1) \quad \ln x = \ln 0.5$ (b) $\ln (x \quad 1) + \ln x = \ln 6$

3. Solve the following exponential equations

(a) $2^x = 4.1$ (b) $3^x = 9.1$ (c) $2 \quad 3^x = 53$ (d) $41 \quad 10 \quad 3^x = 23$

2.3 Properties of the common logarithm

The graph of the common logarithm function $y = \log x$ is similar to the graph of the natural logarithm $y = \ln x$. It is the reflection of the graph of the graph of $y = 10^x$ across the line $y = x$.

The common logarithm has similar properties to the natural logarithm:

If u and v are any positive numbers, and n is any index,

(b) 3

3

Growth & Decay II

3.1 Growth and Decay

We can solve any exponential equation using logarithms.

A population that is growing at a constant rate will have $$P(\mathrm{t}) = P(0)\, e^{rt}$$ members after time t, where • $P(0)$ is the initial population and • r is the constant growth rate per unit time.

Example

The population of China was 850, 000, 000 in 1990 and was growing at the rate of 4% per year.

When did the population reach 1,000,000,000?

Answer

The initial population (in 1990) is $P(0)$ = 850, 000, 000.

Rewrite 4% as a decimal number.

The growth rate is 0.04 per year.

The model is , we need to find when $P(t)$ = 1,000,000,000

Rewrite as an exponential equation.

Take logarithms of both sides.

The population reached 1,000,000,000 in 1994.

Example

The population of China was 850, 000, 000 in 1990 and reached 1,000,000,000 in 1994. If it grew at a constant growth rate, what is this growth rate?

Answer

The initial population (in 1990) is $P(0) = 850, 000, 000$.

The population in 1994 is $P(4) = 1,000,000,000$

The growth model is

4. The ratio of radioactive isotope C^{14} to the regular isotope C^{12} of carbon is fixed in the atmosphere. Living matter breathes in air, and this same ratio of C^{14} to C^{12} is found in all its cells. When it dies and can no take breath in air, the amount of C^{14} begins to decay at a constant rate of 1.24 x10 ^{4}. Th

Example

If a town had an initial population of 1000 and grew at a constant rate, if the population doubled every 30 years, what is the growth rate?

Answer

Similarly, the decay rate of a quantity is usually quite small, and its hard to imagine how fast it decays. Because of this the *half-life* is often quoted instead. The *half-life* of a quantity is the time it takes to halve.

A quanity which is decaying at a constant rate will have the amount

$$Q(t) = Q(0)\, e^{-rt}$$

left after time t, where

- $Q(0)$ is the initial amount and
- r is the constant *decay* rate per unit time.

Example

One kilogram of a radioactive isotope of iodine has a half life of 7.967 years. After this period of time only 500 gm will remain. After a further 7.967 years only 250 gm (ie. half of 500gm) will remain.

A population decaying at a constant decay rate r will be reduced by half

every —— units of time every

Example

One kilogram of a radioactive isotope of iodine decays at a rate of 8.7% per day. I6 of 8.7% perg71uof 8.7% pg7

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Appendix: Answers

Section 1.1

1(a) $\log_2 16 = 4$ (b) $\log_2 1024 = 10$ (c) $\log_2 0.5 = \;\; 1$ (d) $\log_2 1 = 0$

(e) $\log_3 81 = 4$ (f) $\log_4 1024 = 5$ (g) $\log_4 0.5 = \;\; 0.5$ (h) $\log_{10} 1 = 0$

2(a) 2 (b) 4 (c) 0 (d) 1

(e) 1 (f) 2 (g) 0 (h) 2

Section 1.2

1 (a) 0 (b) 0.6931 (c) 1.099 (d) 1.386 (e) 2.197

(f) 0 (g) 0.301 (h) 0.6021 (i) 1.386 (j) 0.5493

2 (a) 0.6931 (b) 0.9163 (c) 0.6931 (d) 1.099